OCR
Oxford Cambridge and RSA

Monday 4 October 2021

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 2 hours 40 minutes

You must have:
- Question Paper Y420/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- harder A Level content

A amber level
- shorter questions (3-6 marks)
- low level problem solving
- harder AS/easier A Level content.

G green level
- short questions (1-3 marks)
- minimal problem solving
- AS/easier A Level content.

E explanation

Section A (31 marks)

Q1: Series (as)
Q2: Calculus (a level)
Q3: Complex Numbers (as)
Q4: Calculus (as)
Q5: Maclaurin Series (a level)
Q6: Matrices (as)

Section B (113 marks)

Q7: Proof (as)
Q8: Algebra (as)
Q9: Matrices (as)
Q10: Complex Numbers (a level)
Q11: Vectors (a level)
Q12: Complex Numbers (as)
Q13: Differential Equations (a level)
Q14: Polar Coordinates (a level)
Q15: Matrices (a level)
Q16: Hyperbolic Functions (a level)
Q17: Differential Equations (a level)

Grade Boundaries

Grade	A*	A	B	C	D	E	U
Mark / 144	102	79	64	49	34	19	0
Scaled/ 180	128	99	80	61	43	24	0

x 1.25

note: the scaled score is added to the scores in the other modules to find an overall grade, not the raw mark

Turn over

Section A (31 marks)

G

1 **(a)** Express $\dfrac{1}{(2r-1)(2r+1)}$ in partial fractions. **[3]**

$$\frac{1}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

← split into 2 fractions with numerators A and B

$$= \frac{A(2r+1) + B(2r-1)}{(2r-1)(2r+1)}$$

$$\Rightarrow A(2r+1) + B(2r-1) = 1$$

$$2Ar + A + 2Br - B = 1$$

$$2A + 2B = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$$

Solve for A and B

$$A - B = 1$$

$$A - (-A) = 1$$

$$\Rightarrow 2A = 1 \Rightarrow A = \tfrac{1}{2} \quad \text{and} \quad B = -\tfrac{1}{2}$$

so $\dfrac{1}{(2r-1)(2r+1)} = \dfrac{1}{2(2r-1)} - \dfrac{1}{2(2r+1)}$

Use values of A and B to write out partial fractions

G

(b) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$, expressing the result as a single fraction. **[4]**

We use method of difference here, using the partial fractions found above.

$$\text{So} \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n} \left(\frac{1}{2(2r-1)} - \frac{1}{2(2r+1)} \right)$$

$$= \frac{1}{2} - \cancel{\frac{1}{6}}$$

$$+ \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}$$

$$\cancel{+} \dots$$

$$+ \cancel{\frac{1}{2(2n-3)}} - \cancel{\frac{1}{2(2n-1)}}$$

$$+ \cancel{\frac{1}{2(2n-1)}} - \frac{1}{2(2n+1)}$$

all of these terms cancel out

$$= \frac{1}{2} - \frac{1}{2(2n+1)} = \frac{2n+1-1}{2(2n+1)} = \frac{2n}{2(2n+1)} = \frac{n}{2n+1}$$

$$\text{hence} \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

G1

2 **In this question you must show detailed reasoning.**

Find the gradient of the curve $y = 6\arcsin(2x)$ at the point with x-coordinate $\frac{1}{4}$. Express the result in the form $m\sqrt{n}$, where m and n are integers. **[4]**

First we need to find the gradient function $\frac{dy}{dx}$.

by chain rule, $y = 6\arcsin u$ $\quad y' = 6 \times \frac{1}{\sqrt{1-u^2}} = \frac{6}{\sqrt{1-u^2}}$

general result from formula book

$u = 2x$ $\quad u' = 2$

so $\frac{dy}{dx} = \frac{6}{\sqrt{1-(2x)^2}} \times 2 = \frac{12}{\sqrt{1-4x^2}}$

We find the gradient of the curve by subbing $\frac{1}{4}$ into $\frac{dy}{dx}$.

multiply by $\frac{\sqrt{3}}{\sqrt{3}}$

gradient $= \left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \frac{12}{\sqrt{1-4(\frac{1}{4})^2}} = \frac{12}{\sqrt{3/4}} = \frac{24}{\sqrt{3}} = 8\sqrt{3}$

$(m = 8,\ n = 3)$

G

3 In this question you must show detailed reasoning.

The complex numbers z_1 and z_2 are given by $z_1 = -2+2\mathrm{i}$ and $z_2 = 2\left(\cos\frac{1}{6}\pi + \mathrm{i}\sin\frac{1}{6}\pi\right)$.

(a) Find the modulus and argument of z_1. **[2]**

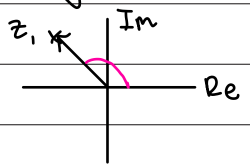
The modulus of a complex number $z = a+bi$ is given by

$|z| = \sqrt{a^2+b^2}$

so $|z_1| = \sqrt{(-2)^2+(2)^2} = 2\sqrt{2}$

The argument of $z = a+bi$ is given by $\arg z = \arctan\frac{b}{a}$

so $\arg z_1 = \arctan\left(\frac{2}{-2}\right) = \arctan(-1) = -\frac{1}{4}\pi$



By drawing a diagram, it is clear we must modify our argument by adding π: $\arg z_1 = -\frac{1}{4}\pi + \pi = \frac{3}{4}\pi$

Modulus = $2\sqrt{2}$

Argument = $\frac{3}{4}\pi$

A

(b) Hence express $\frac{z_1}{z_2}$ in exact modulus-argument form. **[4]**

For the complex numbers z_1 and z_2,

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

hence $\left|\frac{z_1}{z_2}\right| = \frac{2\sqrt{2}}{2} = \sqrt{2}$

$\arg\left(\frac{z_1}{z_2}\right) = \frac{3}{4}\pi - \frac{1}{6}\pi = \frac{7}{12}\pi$

Use rules to find $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$

so $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right)$

Write in form $z = r(\cos\theta + i\sin\theta)$

G1

4 In this question you must show detailed reasoning.

Determine the mean value of $\frac{1}{1+4x^2}$ between $x=-1$ and $x=1$. Give your answer to **3** significant figures. **[4]**

The mean value of a function $f(x)$ between $x = a$ and $x = b$, as given in the formula book is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

So mean value $= \frac{1}{1--1}\int_{-1}^{1} \frac{1}{1+4x^2}\,dx$

Modify into general form

$= \frac{1}{2} \times \frac{1}{4}\int_{-1}^{1} \frac{1}{\frac{1}{4}+x^2}\,dx$

use general result in Formula Book

$= \frac{1}{8}\left[\frac{1}{\frac{1}{2}}\arctan\left(\frac{x}{\frac{1}{2}}\right)\right]_{-1}^{1}$

$= \frac{1}{8} \times 2\left[\arctan(2x)\right]_{-1}^{1}$

$= \frac{1}{4}\left(\arctan 2 - \arctan(-2)\right)$

$= 0.5535\ldots = 0.554$ (3sf)

G

5 **(a)** Use a Maclaurin series to find a quadratic approximation for $\ln(1+2x)$. **[1]**

Given in the formula book, the Maclaurin series for

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

Hence $\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \dots$

$\ln(1+2x) = 2x - 2x^2$

so $\ln(1+2x) \approx 2x - 2x^2$

G

(b) Find the percentage error in using the approximation in part **(a)** to calculate $\ln(1.2)$. **[3]**

Using the approximation above,

$\ln(1.2) = 2(0.1) - 2(0.1)^2 = 0.18$

Percentage error $= \frac{\text{approximation} - \text{real value}}{\text{real value}} \times 100$

so % error $= \frac{0.18 - \ln 1.2}{\ln 1.2} \times 100 = -1.273\dots$

so 1.27% error in $\ln 1.2$

E

(c) Jane uses the Maclaurin series in part **(a)** to try to calculate an approximation for $\ln 3$.

Explain whether her method is valid. **[2]**

From the formula book, $\ln(1+x)$ is valid for $-1 < x \leq 1$

So the approximation in a) is valid for $-1 < 2x \leq 1$

$-\frac{1}{2} < x \leq \frac{1}{2}$

$x = 1$ is used to find $\ln 3$. Since $x = 1$ is not in the interval, the series is not convergent so method is not valid.

A

6 Given that $y = mx$ is an invariant line of the transformation with matrix $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$, determine the possible values of m. **[4]**

To find invariant lines, we solve $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

So $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{array}{l} x + 2y = x' \\ 2x - 2y = y' \end{array}$

So the point (x, mx) on $y = mx$ is mapped to (x', mx') on $y' = mx'$.

$2x - 2y = mx'$

$2x - 2y = m(x + 2y)$ — eliminate x' and y' from the first expression

$2x - 2(mx) = mx + 2m(mx)$

$2x - 2mx = mx + 2m^2x$

$2m^2x + 3mx - 2x = 0$

$(2m^2 + 3m - 2)x = 0$

$(2m - 1)(m + 2)x = 0$

hence $m = \frac{1}{2}$, $m = -2$

Section B (113 marks)

A

7 Prove that $\sum_{r=1}^{n} \frac{r}{2^{r-1}} = 4 - \frac{n+2}{2^{n-1}}$ for all $n \geqslant 1$. **[6]**

Step one: base case

When $n = 1$, $LHS = \sum_{r=1}^{1} \frac{r}{2^{r-1}} = \frac{1}{2^{1-1}} = 1$

$RHS = 4 - \frac{1+2}{2^{1-1}} = 4 - 3 = 1$ $\therefore$ true for $n=1$

Step two: assumption

Assume true for $n=k$, so $\sum_{r=1}^{k} \frac{r}{2^{r-1}} = 4 - \frac{k+2}{2^{k-1}}$

Step three: inductive step

Using the assumed result for $n=k$,

$$\sum_{r=1}^{k+1} \frac{r}{2^{r-1}} = \sum_{r=1}^{k} \frac{r}{2^{r-1}} + \frac{k+1}{2^{k+1-1}}$$

$$= 4 - \frac{k+2}{2^{k-1}} + \frac{k+1}{2^k}$$

$$= 4 - \frac{1}{2^k}\left(\frac{k+2}{1/2} - k - 1\right)$$

$$= 4 - \frac{1}{2^k}(2k + 4 - k - 1)$$

$$= 4 - \frac{1}{2^k}(k+3)$$

$$= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}$$ $\therefore$ true for $n = k+1$.

Step four: conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

G

8 The equation $4x^4 - 4x^3 + px^2 + qx - 9 = 0$, where p and q are constants, has roots α, $-\alpha$, β and $\frac{1}{\beta}$.

(a) Determine the exact roots of the equation. **[5]**

Considering the sums of roots equations should help find the roots.

we know that $\sum \alpha = -\frac{b}{a}$ (sum of roots)

hence $\alpha - \alpha + \beta + \frac{1}{\beta} = -\frac{-4}{4}$

$$\beta + \frac{1}{\beta} = 1$$

$$\beta^2 - \beta + 1 = 0$$

$$\beta = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{1 \pm i\sqrt{3}}{2}$$

and $\frac{1}{\beta} = \frac{1 \mp i\sqrt{3}}{2}$

also $\sum \alpha\beta\gamma\delta = \frac{e}{a}$ (product of roots)

hence $\alpha \times -\alpha \times \beta \times \frac{1}{\beta} = \frac{(-9)}{4}$

$$-\alpha^2 = -\frac{9}{4}$$

$$\alpha^2 = \frac{9}{4}$$

$\alpha = \pm\frac{3}{2}$ also $-\alpha = \mp\frac{3}{2}$

hence roots are $\frac{3}{2}$, $-\frac{3}{2}$, $\frac{1+i\sqrt{3}}{2}$, $\frac{1-i\sqrt{3}}{2}$

Note: look carefully at the equation to see which sum's of roots expressions to consider. eg. In this Q it would not make sense to use $\sum \alpha\beta$ and $\sum \alpha\beta\gamma$ as we do not * yet * know the values of p and q.

G1

(b) Determine the values of p and q. **[4]**

We can now use $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$ as we know what the roots are.

we know that $\Sigma\alpha\beta = \frac{c}{a}$ (sum of products of 2 roots)

$$\frac{3}{2}\left(-\frac{3}{2}\right) + \frac{3}{2}\left(\frac{1+i\sqrt{3}}{2}\right) + \frac{3}{2}\left(\frac{1-i\sqrt{3}}{2}\right) - \frac{3}{2}\left(\frac{1+i\sqrt{3}}{2}\right) - \frac{3}{2}\left(\frac{1-i\sqrt{3}}{2}\right) + \left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right) = \frac{p}{4}$$

$$\Rightarrow \frac{p}{4} = -\frac{5}{4}$$

$$\Rightarrow p = -5$$

We also know that $\Sigma\alpha\beta\gamma = -\frac{d}{a}$ (sum of product of 3 roots)

$$\frac{3}{2}\left(-\frac{3}{2}\right)\left(\frac{1+i\sqrt{3}}{2}\right) + \frac{3}{2}\left(-\frac{3}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right) + \frac{3}{2}\left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right) - \frac{3}{2}\left(\frac{1+i\sqrt{3}}{2}\right)\left(\frac{1-i\sqrt{3}}{2}\right) = -\frac{q}{4}$$

$$\Rightarrow -\frac{q}{4} = -\frac{9}{4}$$

$$q = 9$$

hence $p = -5$, $q = 9$

$p = -5$

$q = 9$

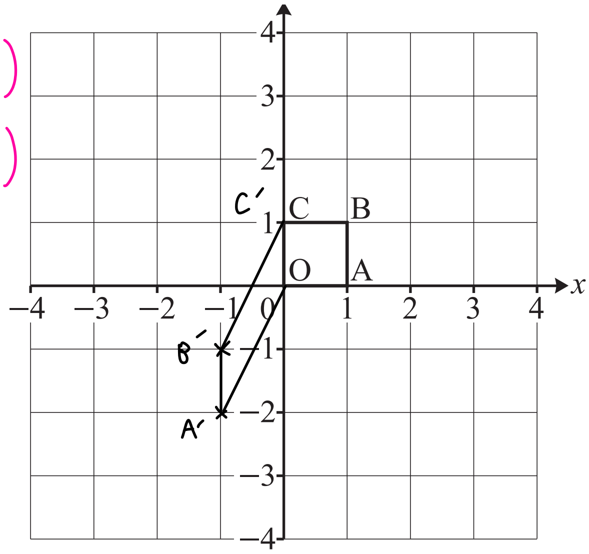
9 The transformation T of the plane has associated matrix **M**, where $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$.

G1

(a) On the grid in the Printed Answer Booklet, plot the image OA′B′C′ of the unit square under the transformation T.

$$\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ -2 & -1 & 1 \end{pmatrix}$$

So $A' = (-1, -2)$
$B' = (-1, -1)$
$C' = (0, 1)$



G1

(b) (i) Calculate the value of det **M**. **[1]**

$\det M = -1(1) - 0(-2) = -1$

E

(ii) Explain the significance of the value of det **M** in relation to the image OA′B′C′. **[2]**

The magnitude of det M is 1, the area scale factor of an object is 1. So area is preserved.
det M < 0 So orientation is reversed.

G1

(c) T is equivalent to a sequence of two transformations of the plane.

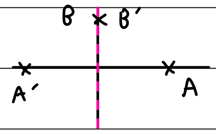
(i) Specify fully **two** transformations equivalent to T. **[3]**

① Reflection in the y-axis.
then
② Shear, y-axis fixed, with $(-1, 0)$ mapped to $(-1, -2)$.

G1

(ii) Use matrices to verify your answer. **[3]**

Reflection: $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



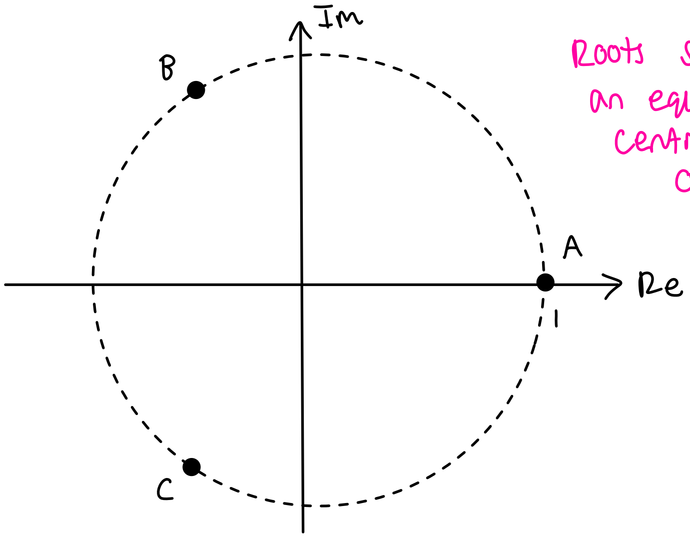
Shear: $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

The transformation matrix that represents a reflection then shear is BA.

$BA = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = M$ as required

G

10 **(a)** Show on an Argand diagram the points representing the three cube roots of unity. **[2]**



Roots should form an equilateral triangle centred at the origin.

One vertex is at $(1, 0)$ as $z = 1$ is the only real root.

A

(b) **(i)** Find the exact roots of the equation $z^3 - 1 = \sqrt{3}\,\mathrm{i}$, expressing them in the form $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta < \pi$. **[5]**

So $z^3 = 1 + \sqrt{3}i \rightarrow |1 + \sqrt{3}i| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$

$\arg(1 + \sqrt{3}i) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Find modulus $r^3 = 2 \Rightarrow r = \sqrt[3]{2}$

and arguments by De Moivre's, $\arg(z_1) = \frac{\pi/3}{3} = \frac{\pi}{9}$

$\arg(z_2) = \frac{\frac{\pi}{3} + 2\pi}{3} = \frac{7}{9}\pi \quad \arg(z_3) = \frac{\frac{\pi}{3} - 2\pi}{3} = -\frac{5}{9}\pi$

so roots are $\sqrt[3]{2}\,e^{\frac{\pi}{9}i}$, $\sqrt[3]{2}\,e^{\frac{7}{9}\pi i}$, $\sqrt[3]{2}\,e^{-\frac{5}{9}\pi i}$

A

(ii) The points representing the cube roots of unity form a triangle Δ_1. The points representing the roots of the equation $z^3 - 1 = \sqrt{3}\,\mathrm{i}$ form a triangle Δ_2.

State a sequence of two transformations that maps Δ_1 onto Δ_2. **[2]**

The root 1 is mapped to $\sqrt[3]{2}\,e^{\frac{\pi}{9}i}$

So ① enlargement by $\sqrt[3]{2}$

② rotation about origin by $\frac{\pi}{9}$ rads. or $20°$.

A

(iii) The three roots in part **(b)(i)** are z_1, z_2 and z_3.

By simplifying $z_1+z_2+z_3$, verify that the sum of these roots is zero. **[2]**

$z_1 + z_2 + z_3 = \sqrt[3]{2}\, e^{-\frac{5}{9}\pi i} + \sqrt[3]{2}\, e^{\frac{\pi}{9} i} + \sqrt[3]{2}\, e^{\frac{7}{9}\pi i}$

This forms a geometric series,

$a = \sqrt[3]{2}\, e^{-\frac{5}{9}\pi i}$, $r = e^{\frac{2\pi}{3} i}$, summed to $n=3$.

So $z_1 + z_2 + z_3 = \dfrac{\sqrt[3]{2}\, e^{-\frac{5}{9}\pi i}\left(1 - \left(e^{\frac{2\pi}{3} i}\right)^3\right)}{1 - e^{\frac{2\pi}{3} i}}$

$= \dfrac{\sqrt[3]{2}\, e^{-\frac{5}{9}\pi i}\left(1 - e^{2\pi i}\right)}{1 - e^{\frac{2\pi}{3} i}}$ $(*)$

in mod-arg form, $e^{2\pi i} = \cos 2\pi + i \sin 2\pi$

$= 1 + i(0) = 1.$

hence as $1-1=0$, the numerator of $(*)$ is 0

$\Rightarrow z_1 + z_2 + z_3 = 0$

A

(iv) Hence show that $\sin 20° + \sin 140° = \sin 100°$. **[2]**

Since $z_1 + z_2 + z_3 = 0$, the imaginary part of the sum of the roots must be zero.

Hence

$\text{Im}\left(\sqrt[3]{2}\, e^{-\frac{5}{9}\pi i} + \sqrt[3]{2}\, e^{\frac{\pi}{9} i} + \sqrt[3]{2}\, e^{\frac{7}{9}\pi i}\right) = 0$

$\text{Im}\Big(\sqrt[3]{2}\left(\cos\left(-\tfrac{5}{9}\pi\right) + i\sin\left(-\tfrac{5}{9}\pi\right)\right)$

$+ \sqrt[3]{2}\left(\cos\left(\tfrac{\pi}{9}\right) + i\sin\left(\tfrac{\pi}{9}\right)\right)$

$+ \sqrt[3]{2}\left(\cos\left(\tfrac{7\pi}{9}\right) + i\sin\left(\tfrac{7\pi}{9}\right)\right)\Big) = 0$

$\sqrt[3]{2}\sin\left(-\tfrac{5}{9}\pi\right) + \sqrt[3]{2}\sin\left(\tfrac{\pi}{9}\right) + \sqrt[3]{2}\sin\left(\tfrac{7\pi}{9}\right) = 0$

$-\sin\left(\tfrac{5}{9}\pi\right) + \sin\left(\tfrac{\pi}{9}\right) + \sin\left(\tfrac{7\pi}{9}\right) = 0$

$-\sin 100° + \sin 20° + \sin 140° = 0$

$\Rightarrow \sin 20° + \sin 140° = \sin 100$ //

G1

11 (a) Given that $\mathbf{u} = \lambda\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, find the following, giving your answers in terms of λ.

(i) $\mathbf{u}.\mathbf{v}$ **[1]**

$$\underline{u} \cdot \underline{v} = \begin{pmatrix} \lambda \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \lambda(1) + 1(2) - 3(-2) = \lambda + 8$$

hence $\underline{u} \cdot \underline{v} = \lambda + 8$

G1

(ii) $\mathbf{u} \times \mathbf{v}$ **[2]**

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \lambda & 1 \\ \underline{j} & 1 & 2 \\ \underline{k} & -3 & -2 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix} - \underline{j}\begin{vmatrix} \lambda & 1 \\ -3 & -2 \end{vmatrix} + \underline{k}\begin{vmatrix} \lambda & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \underline{i}(-2+6) - \underline{j}(-2\lambda+3) + \underline{k}(2\lambda-1)$$

$$= 4\underline{i} + (2\lambda-3)\underline{j} + \underline{k}(2\lambda-1)$$

G1

(b) Hence determine

(i) the acute angle between the planes $2x+y-3z=10$ and $x+2y-2z=10$, **[3]**

The angle between the planes is the angle between the normals of the planes.

Recall that $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$

From a) i., $\underline{u} \cdot \underline{v} = 2 + 8 = 10$

$$\text{so } \theta = \arccos\left(\frac{10}{\sqrt{2^2+1^2+3^2}\sqrt{1^2+2^2+2^2}}\right) = \arccos\left(\frac{5\sqrt{14}}{21}\right) = 27.01\ldots$$

$$= 27.0°$$

A

(ii) the shortest distance between the lines $\dfrac{x-3}{3} = \dfrac{y}{1} = \dfrac{z-2}{-3}$ and $\dfrac{x}{1} = \dfrac{y-4}{2} = \dfrac{z+2}{-2}$, giving your answer as a multiple of $\sqrt{2}$. **[3]**

This distance is given in the FB as $\left|\dfrac{\underline{d}_1 \times \underline{d}_2}{|\underline{d}_1 \times \underline{d}_2|} \cdot (a_1 - a_2)\right|$

$$\underline{r}_1 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} \text{ and } \underline{r}_2 = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$\underline{d}_1 \times \underline{d}_2 = \underline{u} \times \underline{v}$ if $\lambda = 3$, so $\underline{d}_1 \times \underline{d}_2 = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$

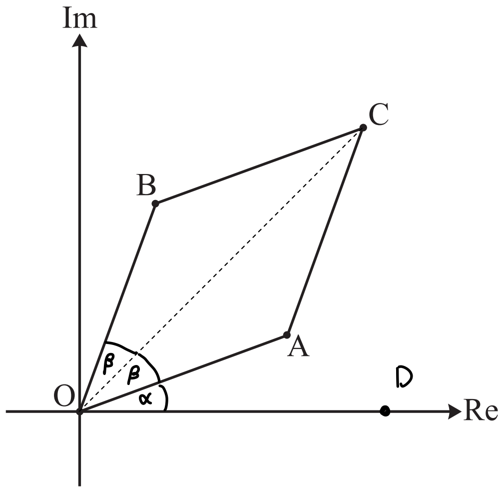
$|\underline{d}_1 \times \underline{d}_2| = \sqrt{4^2 + 3^2 + 5^2} = 5\sqrt{2}$, $\quad a_1 - a_2 = \begin{pmatrix} 3 \\ -4 \\ 4 \end{pmatrix}$

$$\text{so distance} = \frac{1}{5\sqrt{2}}\left|\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 4 \end{pmatrix}\right| = \frac{1}{5\sqrt{2}}(20) = \frac{4}{\sqrt{2}}\left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} \text{ units}$$

A

12 **Fig. 12** shows a rhombus OACB in an Argand diagram. The points A and B represent the complex numbers z and w respectively. Prove that $\arg(z+w) = \frac{1}{2}(\arg z + \arg w)$.



Let $\angle DOA = \alpha$ and $\angle AOC = \beta$

Recall that the sum of two complex number's creates a parallelogram when plotted on an argand diagram.

Hence the point C is represented by $z + w$.

So $\arg(z+w) = \alpha + \beta$

Since OC bisects the parallelogram, $\angle BOC = \beta$

$\arg z = \angle DOA = \alpha$

$\arg w = \angle DOB = \alpha + 2\beta$

so $\arg z + \arg w = \alpha + (\alpha + 2\beta) = 2\alpha + 2\beta$

$= 2(\alpha + \beta)$

$= 2\arg(z+w)$

hence $2\arg(z+w) = \arg z + \arg w$

$\arg(z+w) = \frac{1}{2}(\arg z + \arg w)$

R

13 Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 2e^x$. **[7]**

First we need the complementary solution.

Auxilliary Equation: $\lambda^2 + 2\lambda - 3 = 0$

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda = 1, \quad \lambda = -3$

Since we have two real roots, $y = Ae^x + Be^{-3x} + p(x)$

The particular integral would usually have the form $p(x) = ke^x$.

However, as we already have e^x in the complementary solution, we need to multiply through by x.

So $p(x) = kxe^x$

$p'(x) = kxe^x + ke^x$

$p''(x) = kxe^x + ke^x + ke^x$

$= kxe^x + 2ke^x$

We now sub these into our DE to find k.

$kxe^x + 2ke^x + 2(kxe^x + ke^x) - 3(kxe^x) = 2e^x$

$kxe^x + 2ke^x + 2kxe^x + 2ke^x - 3kxe^x = 2e^x$

$4ke^x = 2e^x$

$4k = 2 \Rightarrow k = \frac{1}{2}$

Hence the general solution to the DE is

$y = Ae^x + Be^{-3x} + \frac{1}{2}xe^x$

R

14 A curve has polar equation $r = a(\cos\theta + 2\sin\theta)$, where a is a positive constant and $0 \leqslant \theta \leqslant \pi$.

(a) Determine the polar coordinates of the point on the curve which is furthest from the pole. **[7]**

We want to maximise r. This occurs when $\frac{dr}{d\theta} = 0$.

$$r = a(\cos\theta + 2\sin\theta)$$

$$\frac{dr}{d\theta} = a(-\sin\theta + 2\cos\theta)$$

$$\frac{dr}{d\theta} = 0 \Rightarrow a(-\sin\theta + 2\cos\theta) = 0$$

$$a \neq 0, \text{ so } -\sin\theta + 2\cos\theta = 0$$

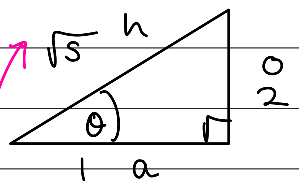
$$-\tan\theta + 2 = 0$$

$$\tan\theta = 2$$

$$\theta = \arctan 2 = 1.107\ldots$$

$$= 1.11 \text{ rad}$$

Now we need the corresponding value of r.



found using pythagoras

$$\tan\theta = 2 \rightarrow \sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{1}{\sqrt{5}}$$

$$\text{hence } r = a\left(\frac{1}{\sqrt{5}} + 2\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$= a\left(\frac{5}{\sqrt{5}}\right)$$

$$= a\sqrt{5}$$

hence point furthest from pole is $(a\sqrt{5}, 1.11)$

A

(b) (i) Show that the curve is a circle whose radius should be specified. **[6]**

We know $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$

so $r = a(\cos\theta + 2\sin\theta)$

$r^2 = ar\cos\theta + 2ar\sin\theta$

$r^2 = ax + 2ay$

$x^2 + y^2 = ax + 2ay$

$x^2 - ax + y^2 - 2ay = 0$

$\left(x - \frac{a}{2}\right)^2 + (y-a)^2 = \left(\frac{a}{2}\right)^2 + (a)^2$

$\left(x - \frac{a}{2}\right)^2 + (y-a)^2 = \frac{5}{4}a^2$

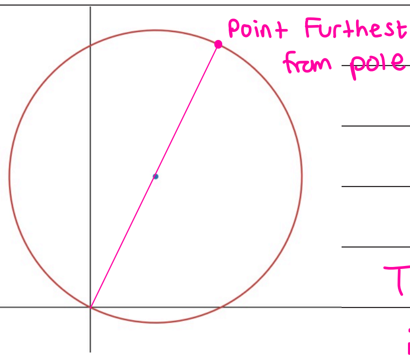
This is an equation of a circle.

It's radius is $\sqrt{\frac{5}{4}a^2} = \frac{\sqrt{5}}{2}a^2$

A

(ii) Write down the polar coordinates of the centre of the circle. **[1]**

centre $= \left(\frac{\sqrt{5}}{2}, 1.11\right)$



The centre has the same value of θ as the point furthest from the pole.

The value of $r = \frac{\sqrt{5}}{2}$ as this is the radius of the circle.

R

15 The equations of three planes are

$$-4x+ky+7z=4,$$
$$x-2y+5z=l,$$
$$2x+3y+z=2.$$

Given that the planes form a sheaf, determine the values of k and l. **[6]**

Let's write this in tabular form.

$$\underbrace{\begin{pmatrix} -4 & k & 7 \\ 1 & -2 & 5 \\ 2 & 3 & 1 \end{pmatrix}}_{\hookrightarrow M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ l \\ 2 \end{pmatrix}$$

The planes will only form a sheaf if they do not intersect at a unique point.

So $\det M = 0$

$$\det M = -4\begin{vmatrix} -2 & 5 \\ 3 & 1 \end{vmatrix} - k\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} + 7\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= -4(-2-15) - k(1-10) + 7(3+4)$$
$$= 68 + 9k + 49$$
$$= 117 + 9k$$

So $117 + 9k = 0$

$$9k = -117$$
$$k = -13$$

We now need to eliminate a variable, say x, and analyse the consistency of these equations.

Let $\Pi_1: -4x - 13y + 7z = 4$

$\Pi_2: x - 2y + 5z = l$

$\Pi_3: 2x + 3y - z = 2$

15 (continued)

$\Pi_1 + 2\Pi_3 : -4x - 13y + 7z + 4x + 6y - 2z = 4 + 4$

$-7y + 9z = 8$

$\Pi_3 - 2\Pi_2 : 2x + 3y - z - 2x + 4y - 10z = 2 - 2l$

$7y - 9z = 2 - 2l$

$-7y + 9z = 2l - 2$

For these planes to form a sheaf, these two equations must be consistent.

Hence $2l - 2 = 8$

$2l = 10$

$l = 5$

So $k = -13$ and $l = 5$

$k = -13$

$l = 5$

G

16 (a) Show using exponentials that $\cosh 2u = 1 + 2\sinh^2 u$. **[4]**

LHS: $\cosh 2u = \dfrac{e^{2u} + e^{-2u}}{2}$ (using $\cosh x = \dfrac{e^x + e^{-x}}{2}$)

RHS: $1 + 2\sinh^2 u = 1 + 2\left(\dfrac{e^u - e^{-u}}{2}\right)^2$

$= 1 + \dfrac{e^{2u} - 1 - 1 + e^{-2u}}{2}$ (using $\sinh x = \dfrac{e^x - e^{-x}}{2}$)

$= \dfrac{e^{2u} + e^{-2u}}{2} = \text{LHS}$ as required

R

(b) Show that $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4+x^2}}\,dx = 2\sqrt{2} - 2\ln(1+\sqrt{2})$. **[10]**

Seing as part a) involved hyperbolics, it is very likely we need a hyperbolic substitution. Look at the formula book, a $\frac{1}{\sqrt{x^2+a^2}}$ form involves $\sinh x$.

Hence let $x = 2\sinh u$

$\dfrac{dx}{du} = 2\cosh u \Rightarrow dx = 2\cosh u \, du$

when $x = 2$, $u = \operatorname{arsinh}\left(\frac{2}{2}\right) = \operatorname{arsinh} 1$

$= \ln\left(1 + \sqrt{1^2+1}\right)$

$= \ln\left(1+\sqrt{2}\right)$

when $x = 0$, $u = \operatorname{arsinh}\left(\frac{0}{2}\right) = \operatorname{arsinh} 0 = 0$

So we now have $\displaystyle\int_0^{\ln(1+\sqrt{2})} \frac{(2\sinh u)^2}{\sqrt{4+(2\sinh u)^2}} \times 2\cosh u \, du$

$\displaystyle = \int_0^{\ln(1+\sqrt{2})} \frac{4\sinh^2 u}{\sqrt{4+4\sinh^2 u}} \, 2\cosh u \, du$

$\displaystyle = \frac{8}{\sqrt{4}} \int_0^{\ln(1+\sqrt{2})} \frac{\sinh^2 u \cosh u}{\sqrt{1+\sinh^2 u}} \, du$

(answer space continued on next page)

16(b)	(continued)

$$= 4\int_0^{\ln(1+\sqrt{2})} \frac{\sinh^2 u \cosh u}{\sqrt{\cosh^2 u}}\, du$$ Using $\cosh^2 u = 1 + \sinh^2 u$

$$= 4\int_0^{\ln(1+\sqrt{2})} \sinh^2 u \times \frac{\cancel{\cosh u}}{\cancel{\cosh u}}\, du$$

$$= 4\int_0^{\ln(1+\sqrt{2})} \sinh^2 u\, du$$ Using part a)

$$= 4\int_0^{\ln(1+\sqrt{2})} \frac{1}{2}(\cosh 2u - 1)\, du$$

$$= 2\int_0^{\ln(1+\sqrt{2})} \cosh 2u - 1\, du$$ Using $\int \cosh ku\, du = \frac{1}{k}\sinh ku + c$

$$= 2\left[\frac{1}{2}\sinh 2u - u\right]_0^{\ln(1+\sqrt{2})}$$

$$= \sinh(2\ln(1+\sqrt{2})) - 2\ln(1+\sqrt{2}) - (\sinh 0 - 0)$$ Using $\sinh u = \frac{1}{2}(e^u - e^{-u})$

$$= \frac{1}{2}\left(e^{2\ln(1+\sqrt{2})} - e^{-2\ln(1+\sqrt{2})}\right) - 2\ln(1+\sqrt{2})$$

$$= \frac{1}{2}\left((e^{\ln(1+\sqrt{2})})^2 - (e^{\ln(1+\sqrt{2})})^{-2}\right) - 2\ln(1+\sqrt{2})$$ $e^{\ln(f(x))} = f(x)$

$$= \frac{1}{2}\left((1+\sqrt{2})^2 - (1+\sqrt{2})^{-2}\right) - 2\ln(1+\sqrt{2})$$

$$(1+\sqrt{2})^{-2} = \frac{1}{1+2\sqrt{2}+2} = \frac{1}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$ (rationilising denominator)

$$= \frac{3-2\sqrt{2}}{1} = 3 - 2\sqrt{2}$$

so integral $= \frac{1}{2}\left(3 + 2\sqrt{2} - (3 - 2\sqrt{2})\right) - 2\ln(1+\sqrt{2})$

$= \frac{1}{2}(4\sqrt{2}) - 2\ln(1+\sqrt{2})$

$= 2\sqrt{2} - 2\ln(1+\sqrt{2})$ as required

17 In a chemical process, a vessel contains 1 litre of pure water. A liquid chemical is then passed into the top of the vessel at a constant rate of a litres per minute and thoroughly mixed with the water. At the same time, the resulting mixture is drawn from the bottom of the vessel at a constant rate of b litres per minute. You may assume that the chemical mixes instantly and uniformly with the water. After t minutes, the mixture in the vessel contains x litres of the chemical.

G1

(a) **(i)** Show that the proportion of chemical present in the vessel after t minutes is $\frac{x}{1+(a-b)t}$. **[2]**

When $t = 0$, there is 1 litre of pure water.
a liters of chemical is added per minute, and
b liters is taken out per minute.

Hence total in vessel is $1 + at - bt$
$= 1 + (a-b)t$

So proportion is $\frac{\text{chemical}}{\text{total}} = \frac{x}{1+(a-b)t}$

G1

(ii) Hence show that $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{bx}{1+(a-b)t} = a$. **[2]**

The rate of chemical in is a liters/hour.
The rate of chemical out is $\frac{bx}{1+(a-b)t}$ liters/hour.

Since $\frac{dx}{dt}$ is the rate of change of the chemical in the mixture,

$$\frac{dx}{dt} = a - \frac{bx}{1+(a-b)t}$$

$$\frac{dx}{dt} + \frac{bx}{1+(a-b)t} = a$$

A

(b) First, consider the case where $b = a$.

(i) Solve the differential equation to find x in terms of a and t. **[4]**

If $b = a$, the DE is $\frac{dx}{dt} + \frac{ax}{1+(a-a)t} = a$.

$$\frac{dx}{dt} + ax = a$$

$$\Rightarrow \frac{dx}{dt} = a(1-x)$$

Via seperation of variable

$$\int \frac{1}{1-x}\,dx = \int a\,dt$$

$$-\ln|1-x| = at + c$$

When $t=0$, $a=0$. so $-\ln(1) = a(0) + c$

$$c = 0$$

hence $-\ln|1-x| = at$ write in required form

$$1 - x = e^{-at} \Rightarrow x = 1 - e^{-at}$$

A

(ii) Given that after 1 minute the vessel contains equal amounts of water and chemical, find the rate of inflow of chemical. **[2]**

at $t=1$, total in container $= 1 + a - a = 1$.

So at $t=1$, $x = \frac{1}{2}$.

$$\frac{1}{2} = 1 - e^{-a(1)}$$

solve for a (rate of inflow of chemical)

$$e^{-a} = \frac{1}{2}$$

$$-a = \ln\frac{1}{2}$$

$$-a = -\ln 2$$

$$a = \ln 2 = 0.6931... = 0.693 \text{ (3 sf)}$$

so rate of inflow is 0.693 liters/minute

E

(c) Now consider the case where $b = 2a$.

(i) Explain why the differential equation in part **(a)(ii)** is now invalid for $t \geqslant \frac{1}{a}$. **[1]**

at $t = \frac{1}{a}$, the amount of mixture left is

$$1 + a\left(\frac{1}{a}\right) - 2a\left(\frac{1}{a}\right) = 0$$

Hence at $t = \frac{1}{a}$ there is no mixture left, so after this the amount of mixture is negative. so $t \geqslant \frac{1}{a}$.

R

(ii) Find the maximum amount of chemical in the vessel. **[9]**

First we need to solve the DE to find x in terms of t.

The DE is now $\frac{dx}{dt} + \frac{2ax}{1-at} = a$

This is a linear first order DE, so can be solved using an integrating factor method.

$$I(t) = e^{\int \frac{2a}{1-at}\,dt} = e^{2\int \frac{a}{1-at}\,dt}$$

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$$= e^{-2\ln|1-at|}$$

$e^{\ln(f(x))} = f(x)$

$$= (1-at)^{-2}$$

So multiplying through by $I(t)$,

$\times I(t)$: $(1-at)^{-2}\frac{dx}{dt} + (1-at)^{-2} \times \frac{2ax}{1-at} = a(1-at)^{-2}$

$$(1-at)^{-2}\frac{dx}{dt} + 2ax(1-at)^{-3} = a(1-at)^{-2}$$

The LHS is implicit product rule. $u = x$ $\quad u' = \frac{dx}{dt}$

$v = (1-at)^{-2}$ $\quad v' = 2a(1-at)^{-3}$

So $\frac{d}{dt}\left(x(1-at)^{-2}\right) = a(1-at)^{-2}$

$$x(1-at)^{-2} = \int a(1-at)^{-2}\,dt$$

by recognition

$$x(1-at)^{-2} = (1-at)^{-1} + c$$

at $t=0$, $x=0$: $0(1-a(0))^{-2} = (1-a(0))^{-1} + c$

$$0 = 1 + c$$

$$c = -1$$

(answer space continued on next page)

17(c)(ii)	(continued)
	hence $x(1-at)^{-2} = (1-at)^{-1} - 1$
	$\times (1-at)^2$: $x = 1 - at - (1-at)^2$
	$x = 1 - at - (1 - 2at + a^2t^2)$
	$x = at - a^2t^2$
	To solve the question we need to find the
	maximum value of x. This occurs when
	$\frac{dx}{dt} = 0$.
	$\frac{dx}{dt} = a - 2ta^2$
	$a - 2ta^2 = 0$
	$\Rightarrow 2ta^2 = a$
	$t = \frac{1}{2a}$ Sub in to find x
	when $t = \frac{1}{2a}$, $x = a\left(\frac{1}{2a}\right) - a^2\left(\frac{1}{2a}\right)^2$
	$= \frac{1}{2} - \frac{1}{4}$
	$= \frac{1}{4}$
	So the maximum amount of chemical is 0.25 L.

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).